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Anharmonic theory of partially ordered and disordered phases in magnets with easy-plane single-ion anisotropy

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Abstract. Dynamic properties of collinear phases realised in magnets with easy-plane single-ion anisotropy in an external magnetic field perpendicular to an 'easy plane' are studied. A phase diagram characterised by the existence of a cascade of reorientational phase transitions induced by the field and the single-ion anisotropy is constructed. The spectrum of collective excitations, their damping, scattering amplitudes, critical fields at low temperatures, position of multicritical points, and critical behaviour peculiarities at $T = 0$ in the vicinity of the second-order phase transition lines and of the multicritical points are investigated.

1. Introduction

Recent investigations have shown that the behaviour of magnets with single-ion anisotropy (SA) differs substantially from that of exchange-anisotropy magnets of the same symmetry. In the case of easy-plane ferromagnets (FM) this difference is so large that it qualitatively changes the phase diagram. Namely, in the SA case there can be a cascade of the $2S$ second-order field-induced phase transitions [1] whereas in the exchange-anisotropy case only one phase transition is possible (S is the site spin value). The phases in the cascade undergo transformation from ferromagnetically disordered phase to saturated ferromagnetic phase via partially ordered ones. A schematic representation of the T–h phase diagram of a ferromagnet with SA for some value of D greater than $D = JS(S + 1)$ is given in figure 1 and the D–h phase diagram at $T = 0$ for the case $S = 2$ is given in figure 2. (Here T is the temperature, h is the magnetic field, D is the SA constant and J is the exchange integral; the phase diagram is constructed using the results of [1].) The phase diagrams are characterised by the alteration of phases with spontaneously broken symmetry with respect to rotations around the hard axis and the phases with the preserved symmetry, so that at h = const the cascade of the second-order anisotropy-induced phase transitions is realised and at D = const the cascade of the second-order field-induced phase transitions is realised. The specific feature of the phase diagram is the existence of multicritical points (MP), i.e. points $A^{(m)}$ in figure 2. At arbitrary S the general form of the phase diagram at $T = 0$ is preserved, the number of fragments with MP $A^{(m)}$ is $2S - 1$, m acquires the values $S, S - 1, \ldots, -S$. The multicriticality character of the points $A^{(m)}$ is demonstrated in figure 3 where the central fragment of the phase diagram is given in the coordinate system T–D–h (for the case of $S = 1$).

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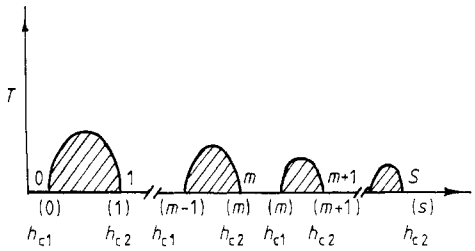

Figure 1. The T–h phase diagram of an easy-plane magnet for the case of integer S and $D > J_0S(S+1)$. Shaded areas are the phases with spontaneously distorted symmetry (non-collinear phases), the unshaded areas are the phases with conserved symmetry (collinear phases). Collinear phases are numbered with number m such that at $T = 0$ exact equality $\langle S^z \rangle = m$ is valid and they are transformed from magnetically ordered phase at $m = 0$ to saturated FM phase at $m = S$ via partially ordered phases.

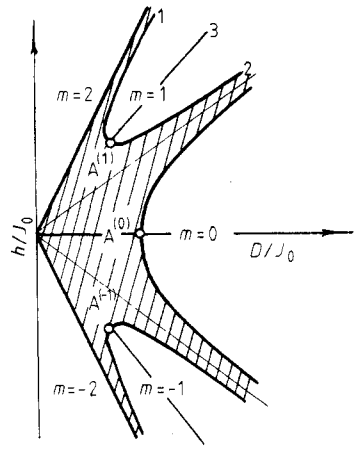

Figure 2. The h–D phase diagram at $T = 0$ for $S = 2$. Lines 1 and 2 are the lines of the second-order reorientational phase transitions between non-collinear (shaded area) and collinear (unshaded area) phases: 1, $h_{c1}^{(m)}(\xi)$; 2, $h_{c2}^{(m)}(\xi)$. Line 3 is the collinear phase symmetry line determined by equality $H = \Delta_2$. $A^{(m)}$ are the multicritical points.

In [1] the properties of non-collinear and collinear phases were investigated in the spin-wave approximation. The purpose of the present paper is the construction of the non-linear low-temperature theory of collinear phases, which takes into account the quasi-particle interaction processes.

For these phases the total magnetic moment $\mathcal{S}^z = (1/N)\Sigma_i S_i^z$ is the motion integral; at $T = 0$ exact equality $\langle S^z \rangle = m$ is valid, and according to this we enumerate the collinear phases with number m taking values $0, 1, \ldots, S$ for integer S and $\frac{1}{2}, \frac{3}{2}, \ldots, S$ for half-integer S.

Thus the aim of our paper is to describe the dynamics of the collinear phase with an arbitrary m at an arbitrary S. In the limiting cases $m = 0$ and $m = S$ the results should coincide with the known results for the non-magnetic phase [2] and isotropic FM [3] (the latter at $D = 0$). The Goldstone theorem and the Adler principle† should be valid in the latter case. Moreover, the kinematic condition should be fulfilled, namely at $S = \frac{1}{2}$ all effects associated with the SA should disappear.

For a description of dynamic properties of these phases, the standard magnetism theory transformations, i.e. Holstein–Primakoff and Maleev–Dyson transformations, are not applicable since they presuppose (i) full FM ordering at $T = 0$ ($\langle S^\alpha \rangle_0 = S$) and (ii) the existence of a single mode of collective excitations described by the Bose operators of these transformations. Neither condition is fulfilled in our case. Therefore, we make use of the special transformation developed in [6], which is the generalisation of the above transformations for the case when there are tensor interactions in the system (SA is the local tensor interaction).

† The term 'Adler principle' means the certain symmetry relations for scattering amplitudes (see for example [4]) which are identical to the Adler theorem for systems with a degenerate vacuum [5]. For example these relations lead to the fact that amplitude tends to zero when the scattered quasi-particle wavevector tends to zero.

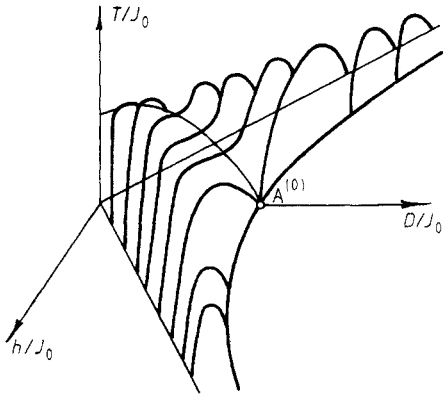

Figure 3. Central fragment of the phase diagram with multicritical point $A^{(0)}$ in T–D–h coordinates (it is given for the case of $S = 1$).

It should be mentioned also that, in accordance with the concept in [6], ordering in any phases realised in systems with SA is described by the order parameter with both vector (FM) and tensor components in the general case. In the case under consideration for the collinear phase numbered with m, the contribution of ferromagnetic components is proportional to m and the net contribution of the tensor components is proportional to $S - m$.

In this paper the spectrum of low-energy excitations, their damping, scattering amplitudes, thermodynamic properties of collinear phases, critical fields at low T, position of multicritical points, and peculiarities of critical behaviour at $T = 0$ in the vicinity of the second-order phase transition points and the multicritical points are determined. The results are obtained in the first approximation on r_0^{-3} for the spectrum and in the second approximation for the thermodynamic functions (r_0 is the exchange interaction radius). The difference in the system behaviour in the cases of ferro- and antiferromagnetic exchange is analysed.

2. Hamiltonian

The Hamiltonian of a ferromagnet with the easy-plane SA in the field perpendicular to the 'easy plane' has the form

$$\mathcal{H} = -\frac{1}{2}\sum_{ij} J_{ij}(\boldsymbol{S}_i\boldsymbol{S}_j) + D\sum_i (S_i^z)^2 - h\sum_i S_i^z \qquad (D > 0). \tag{1}$$

Let us use transformation [6] from the spin and tensor operators to the Bose operators. As for the collinear phases the local coordinate system coincides with the initial ones [1], we pass directly from the spin and tensor operators to the Hubbard ones

$$S^+ = \frac{1}{\sqrt{2}}\sum_{n=-S}^{S} \gamma_S(n)X^{n+1,n} \qquad S^- = \frac{1}{\sqrt{2}}\sum_{n=-S}^{S} \gamma_S(n)X^{n,n+1}$$
$$S^z = \sum_n nX^{nn} \qquad \gamma_S(n) = [(S-n)(S+n+1)]^{1/2} \tag{2}$$

and then to the Bose operators by the formulae

$$X^{p_e p_0} = a_e^+ \left(1 - \sum_{n=1}^{2S} a_n^+ a_n\right) \qquad (X^{p_e p_0})^+ \equiv X^{p_0 p_e} = a_e$$
$$X^{p_e p_e} = a_e^+ a_e \qquad X^{p_e p_{e'}} = a_e^+ a_{e'} \qquad e, e' = 1, \ldots, 2S. \tag{3}$$

Here p_0 is the ion ground level, p_e are the excited levels, $e = 1, 2 \ldots, 2S$. Analysing the Hamiltonian in the molecular-field approximation, one can easily see that for the collinear phase numbered with m the ground level is $|m\rangle$, the nearest excited levels are $|m-1\rangle$ and $|m+1\rangle$, and the next-nearest levels are $|m-2\rangle$ and $|m+2\rangle$. As was shown [1], in the spin-wave approximation there are two modes of collective excitations (α and β modes) with frequencies $\omega_k^{\alpha,\beta}$ which are the superposition of itinerant transitions $|m\rangle \rightarrow |m \pm 1\rangle$; the other excitations remain localised and do not affect the dynamics of the system. One can easily prove that with the anharmonic effects taken into account some of them, namely excitations $|m\rangle \rightarrow |m \pm 2\rangle$, play an essential part: they are involved in inelastic scattering of quasi-particles with frequencies $\omega_k^{\alpha,\beta}$. One can also see that other excitations connected with the transitions to higher levels still describe the subsystem of high-lying localised excitations, which is independent of the subsystem of low-energy quasi-particles and, consequently, produces no effect on the low-temperature dynamics of the system.

Thus while describing the low-temperature dynamics one has to take into account only four operators of the $2S$ Bose operators introduced by the full transformation, namely $a \equiv a_1$, $b \equiv a_2$, associated with the transitions to levels $p_1 = m - 1$, $p_2 = m + 1$, and $c \equiv a_3$, $d \equiv a_4$, associated with the transitions to levels $p_3 = m - 2$, $p_4 = m + 2$. Consequently, in equation (2) sums over n should be restricted to $n = m \pm 1, m \pm 2$.

As a result, the Hamiltonian describing the collinear phase numbered with m acquires the form:

$$\mathcal{H} = E_0 + \mathcal{H}_2 + \mathcal{H}_3 + \mathcal{H}_4 + \mathcal{H}_5 + \mathcal{H}_6$$

where $\mathcal{H}_n$ are the forms of order n with respect to the Bose operators. Passing to the wavevector space one obtains the following quadratic form describing the system dynamics in the Gaussian approximation:

$$\mathcal{H}_2 = \sum_k [(A_k + C_k)a_k^+ a_k + (A_k - C_k)b_k^+ b_k + B_k(a_k b_{-k} + b_{-k}^+ a_k^+) + y_c c_k^+ c_k + y_d d_k^+ d_k] \tag{4}$$

where

$$A_k = D - J_k(S^2 - m^2 + S)/2 \qquad B_k = -J_k \gamma_s(m)\gamma_s(-m)/2$$
$$C_k = h + J_0 m - 2Dm - J_k m/2$$
$$y_c = 4D(1-m) + 2(J_0 m + h) \qquad y_d = 4D(1+m) - 2(J_0 m + h) \tag{5}$$
$$J_k = \sum_{i-j} J_{ij}\, e^{i\mathbf{k}\cdot(\mathbf{R}_i - \mathbf{R}_j)}.$$

The Hamiltonians describing the quasi-particle interaction processes are

$$\begin{aligned}\mathcal{H}_4 = \sum_{1234} \{&(J_{3-1}/2)(a_1^+ b_2^+ a_3 b_4 + b_1^+ a_2^+ b_3 a_4 \\ &- b_1^+ b_2^+ b_3 b_4 - a_1^+ a_2^+ a_3 a_4) + (J_4/2)[\gamma_s^2(m)(b_1^+ b_2^+ b_3 b_4 \\ &+ b_1^+ a_2^+ a_3 b_4) + \gamma_s^2(-m)(a_1^+ a_2^+ a_3 a_4 + a_1^+ b_2^+ b_3 a_4)] \\ &+ (J_1/2)\gamma_s(m)\gamma_s(-m)(a_1^+ b_2^+ a_{-3}^+ a_4 + a_1^+ b_2^+ b_{-3}^+ b_4 \\ &+ b_1^+ a_2^+ a_{-3}^+ a_4 + b_1^+ a_2^+ b_{-3}^+ b_4)\}\delta(1+2-3-4)\end{aligned} \tag{6}$$

and

$$\mathcal{H}_3 = \sum_{123} [V_{bd}(1,2) b_1^+ b_2^+ d_3 + V_{ac}(1,2) a_1^+ a_2^+ c_3 + \text{HC}] \tag{7}$$

where

$$\begin{aligned}V_{bd}(1,2) &= -\tfrac{1}{2}(J_1 + J_2)\gamma_s(m)\gamma_s(m+1) \\ V_{ac}(1,2) &= -\tfrac{1}{2}(J_1 + J_2)\gamma_s(-m)\gamma_s(-m-1).\end{aligned}$$

($\mathcal{H}_5$ and $\mathcal{H}_6$ are not written here since they are not used below; HC is the Hermitian conjugation symbol.)

3. Gaussian approximation for ferromagnetic exchange case ($J > 0$)

In this section and below we pass to the relative values renormalised by $J_0(S^2 + S - m^2)$. In particular, we introduce relative anisotropy, magnetic field and collective excitation spectrum

$$\begin{aligned}&\xi = D/J_0(S^2 + S - m^2) \qquad H = h/J_0(S^2 + S - m^2) \\ &\Omega_k^{\alpha,\beta} = \omega_k^{\alpha,\beta}/J_0(S^2 + S - m^2) \\ &Y_e = y_e/J_0(S^2 + S - m^2) \qquad (e = c, d)\end{aligned} \tag{8}$$

and dimensionless parameters

$$r = \gamma_s(m)\gamma_s(-m)/(S^2 + S - m^2) \qquad F = 1/(S^2 + S - m^2). \tag{9}$$

Hamiltonian (4) is diagonalised by the u–v transformation

$$\begin{aligned}a_k^+ &= u_k \alpha_k^+ + v_k \beta_k \qquad & b_k^+ &= u_k \beta_{-k}^+ + v_k \alpha_{-k} \\ a_k &= u_k \alpha_k + v_k \beta_k^+ \qquad & b_k &= u_k \beta_{-k} + v_k \alpha_{-k}^+.\end{aligned} \tag{10}$$

The spectrum and the functions of the u–v transformation have the form

$$\begin{aligned}&\Omega_k^{\alpha,\beta} = \varepsilon_k \pm [H - \Delta_2 + V(1 - \nu_k)] \\ &\varepsilon_k = (\operatorname{sgn} A_k)\{[\xi - (r+1)\nu_k/2][\xi + (r-1)\nu_k/2]\}^{1/2} \\ &\Delta_2 = m(2\xi - F/2) \qquad \nu_k = \sum_\delta e^{ik\cdot\delta} \qquad V = mF/2\end{aligned} \tag{11}$$

and

$$u_k^2 = \frac{1}{2}\left(\frac{\xi - \nu_k/2}{\varepsilon_k} + 1\right) \qquad v_k^2 = \frac{1}{2}\left(\frac{\xi - \nu_k/2}{\varepsilon_k} - 1\right). \tag{12}$$

The α- and β-mode softening lines define the lines of the second-order phase transitions to the non-collinear phases [1]—lines 1 and 2 in figure 2, which, in this approximation, are described by the formulae

$$H_{c1,c2}^{(m)}(\xi) = \Delta_2 \pm \Delta_1 \qquad (\Delta_1 \equiv \varepsilon_0). \tag{13}$$

Points where $H_{c1}^{(m)} = H_{c2}^{(m)}$ defined by equalities $\Delta_1 = 0, H = \Delta_2$ (points $A^{(m)}$ in phase diagram of figure 2) are the multicritical points (MP) characterised by a peculiar spectrum behaviour, namely, there are two soft modes with the linear dispersion law at small k - $\Omega_k^\alpha = \Omega_k^\beta \sim k$ instead of one soft mode with the quadratic dispersion law at any other point of lines $H_{ci}^{(m)}(\xi)$.

Lines $H = \Delta_2$ (lines 3 in figure 2) are the symmetry lines of the collinear phase with fixed m. At these lines the spectrum at $k = 0$ is degenerate, $\Omega_0^\alpha = \Omega_0^\beta$; at $H > \Delta_2$ the low-lying mode is that with frequency Ω_k^β decreasing with H and at $H < \Delta_2$ that with frequency Ω_k^α increasing with H. Since low-temperature behaviour of thermodynamic functions is defined by the low-energy mode behaviour, particularly by the field dependence of the frequencies, the lines $H = \Delta_2$ are simultaneously the lines separating the regions with qualitatively different behaviour of some macroscopic properties. For example the magnetisation M^z and spin heat capacity, which are connected with the spectrum by formulae [7]

$$M^z(T) - M^z(0) = -\sum_{k\mu} (\partial \omega_k^\mu / \partial h) n(\omega_k^\mu) \qquad C_s(T) = (\partial/\partial T) \sum_{k\mu} \omega_k^\mu n(\omega_k^\mu)$$

($n(x)$ is the Bose factor), are characterised by the following behaviour: M^z increases with T at fixed H from the region $H > \Delta_2$ and decreases with T at H from the region $H < \Delta_2$; C_s decreases with H at $H < \Delta_2$ and increases at $H > \Delta_2$ (T = const).

It can easily be proved that for the phase with $m = S$ equality $r = 0$ is valid and for all other m the value of r varies over a very narrow range: from maximum value $r = 1$ at $m = 0$ and arbitrary S to minimum value $r = 2\sqrt{2/3} \simeq 0.94$ at $m = S - 1$ and $S \to \infty$. Therefore with a good accuracy one can assume†

$$r = \begin{cases} 0 & m = S \\ 1 & m \neq S. \end{cases} \tag{14}$$

Thus for the phases with $m \neq S$ some characteristics acquire universal form

$$\varepsilon_k = [\xi(\xi - \nu_k)]^{1/2} \qquad \Delta_1 = [\xi(\xi - 1)]^{1/2} \qquad \xi_A = 1 \tag{15}$$

which do not depend on m and S but only on relative anisotropy ξ. In equation (15) ξ_A is the multicritical point coordinate. Respectively, the existence region of each phase with $m \neq S$ is $\xi \in (1, \infty)$. Characteristics dependent on the relative magnetic field pre-

† The approximations of this type might have upset relations like the Goldstone theorem and the Adler principle. However, collinear phases with $m \neq S$ for which they are made are the phases with preserved symmetry. Consequently, these exact relationships are not valid. For numerical estimates this approximation is very good and allows one to simplify the results considerably, yielding some universality of properties at various m.

serve their explicit dependence on m and S. For example, the magnetic field at multicritical point $A^{(m)}$ is

$$H_A = m(2 - F/2). \tag{16}$$

It should be noted that far from the MP the excitation spectrum at small wavevector magnitude can be written in a simple form

$$\begin{aligned} &\Omega_k^\alpha = \Delta^\alpha + U^\alpha k^2 && \Omega_k^\beta = \Delta^\beta + U^\beta k^2 \\ &\Delta^\alpha = H - H_{c2}^{(m)} && \Delta^\beta = H_{c1}^{(m)} - H \\ &U^{\alpha,\beta} = \rho(\xi/2\Delta_1 \pm V) && \rho = \lim_{k\to 0}(1 - \nu_k)/k^2. \end{aligned} \tag{17}$$

For the phase with $m = S$ there is only one branch of collective excitations

$$\omega_k^\alpha = h - D(2S - 1) + J_0 S(1 - \nu_k)$$

(since the sums over n in equation (2) are restricted to terms with $n < S$, branch ω_k^β possesses no dispersion). As $u_k^2 = 1$, $v_k^2 = 0$, there are no zero oscillations. (This is quite natural since the zero oscillations in the collinear phases under consideration are the oscillations of the order parameter tensor components.) Owing to the absence of the second excitation branch and, consequently, of the second critical field, the multicritical point is absent too. Therefore the existence region of the phase with $m = S$ is not restricted with respect to anisotropy: $\xi \in (0, \infty)$.

Let us introduce the Green functions by the relationships

$$K_{AB}(k, \tau) = \langle TA_k(\tau)B_k(0)\rangle \qquad (A, B = a, b, c, d)$$

where T is the chronological operator and τ is the imaginary time. Their Fourier transforms over discrete frequency $\omega_n = 2\pi n/\beta$ are

$$\begin{aligned} K_{aa^+}(k, i\omega_n) &= u_k^2/[\beta(\omega_k^\alpha - i\omega_n)] + v_k^2/[\beta(\omega_k^\beta + i\omega_n)] \\ K_{bb^+}(k, i\omega_n) &= u_k^2/[\beta(\omega_k^\beta - i\omega_n)] + v_k^2/[\beta(\omega_k^\alpha + i\omega_n)] \\ K_{a^+b^+}(k, i\omega_n) &= K_{ba}(k, i\omega_n) \\ &= u_k v_k\{1/[\beta(\omega_k^\beta - i\omega_n)] + 1/[\beta(\omega_k^\alpha + i\omega_n)]\} \\ K_{ff^+}(k, i\omega_n) &= 1/[\beta(y_f - i\omega_n)] \qquad f = c, d \qquad (\beta \equiv 1/k_B T). \end{aligned} \tag{18}$$

Let us also define the correlation functions

$$\begin{aligned} n_p^a &\equiv \langle a_p^+ a_p\rangle = u_p^2 n(\omega_p^\alpha) + v_p^2 n(\omega_p^\beta) + v_p^2 \\ n_p^b &\equiv \langle b_p^+ b_p\rangle = u_p^2 n(\omega_p^\beta) + v_p^2 n(\omega_p^\alpha) + v_p^2 \\ \mu_p &\equiv \langle b_p^+ a_{-p}^+\rangle = \langle a_p b_{-p}\rangle = u_p v_p[1 + n(\omega_p^\alpha) + n(\omega_p^\beta)] \end{aligned} \tag{19}$$

with $n(x) = (e^{\beta x} - 1)^{-1}$, which will be necessary for determination of anharmonic corrections.

4. Anharmonic effects: renormalised spectrum, scattering amplitudes, excitation damping and free energy

4.1. Polarisation operator

According to Dyson's equation the Green functions renormalised at the expense of the collective excitation interaction are connected with the polarisation operator. In matrix form this connection is

$$\hat{\tilde{K}}^{-1} = \hat{K}^{-1} - \hat{\Pi} \tag{20}$$

where $\hat{K}$ and $\hat{\tilde{K}}$ are the matrices of spin-wave and renormalised Green functions, respectively, and $\hat{\Pi}$ is the matrix polarisation operator.

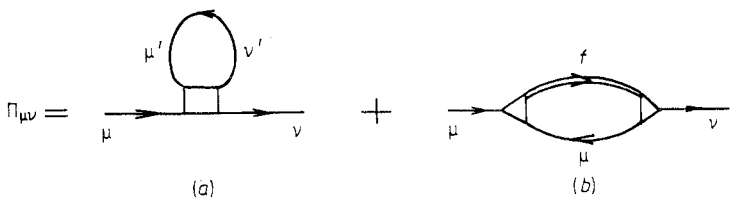

Figure 4. Graphs for polarisation operator components $\Pi_{\mu\nu}$ ($\mu, \nu = a, a^+, b, b^+$) of the first order in r_0^{-3}. Single lines correspond to the Green functions $K_{\mu\nu}$ ($\mu, \nu = a, a^+, b, b^+$); double lines correspond to the Green functions K_{ff^+} ($f = c, d$). In vertices there are trial interactions determined by $\mathcal{H}_4$ and $\mathcal{H}_3$.

It can easily be proved that the account of $\mathcal{H}_4$ in the first-order perturbation theory and of $\mathcal{H}_3$ in the second-order theory yields the values of the polarisation operator components of the same order with respect to r_0^{-3}, namely of the first order, since in both cases there is one summation over an intermediate momentum (see figure 4). Using an explicit form of the spin-wave Green functions (18) and interaction potentials defined by $\mathcal{H}_4$, one gets for

$$\tilde{\Pi}^{(4)}_{\mu\nu} \equiv \frac{\Pi^{(4)}_{\mu\nu}}{J_0(S^2 + S - m^2)}$$

in accordance with figure 4(a)

$$\begin{aligned}\tilde{\Pi}^{(4)}_{aa^+}(k, \mathrm{i}\omega_n)/\beta = -(1/N)\sum_p \{&(\nu_k + \nu_p)(1 - mF)n_p^a + [(\nu_k + \nu_p) \\ &+ (\nu_k - \nu_p)mF]n_p^b + r(\nu_k + 3\nu_p)\mu_p/2 \\ &+ F(n_p^b - n_p^a) - F\nu_{k-p}n_p^a\}\end{aligned} \tag{21}$$

$$\tilde{\Pi}^{(4)}_{a^+b^+}(k, \mathrm{i}\omega_n)/\beta = -(1/N)\sum_p [r(3\nu_k + \nu_p)(n_p^a + n_p^b)/2 + \nu_p\mu_p + F\nu_{k-p}\mu_p]$$

$$\tilde{\Pi}^{(4)}_{ab}(k, \mathrm{i}\omega_n)/\beta = -(1/N)\sum_p (\nu_k\mu_p + F\nu_{k-p}\mu_p).$$

(An expression for $\bar{\Pi}^{(4)}_{bb^+}$ can be obtained from that for $\bar{\Pi}^{(4)}_{aa^+}$ by substitutions $n^a_p \to n^b_p$, $n^b_p \to n^a_p$, $m \to -m$.) Additional terms from $\mathcal{H}_3$ in accordance with figure 4(b) have the form

$$\begin{aligned}
\bar{\Pi}^{(3)}_{aa^+}(k, \mathrm{i}\omega_n) &= (\beta/N)\sum_p [u^2_{p-k}(n^c - n^\alpha_{p-k})/(\Omega^\alpha_{p-k} - Y_c + \mathrm{i}\omega_n) \\
&\quad + v^2_{p-k}(1 + n^c + n^\beta_{p-k})/(\Omega^\beta_{p-k} + Y_c - \mathrm{i}\omega_n)]\bar{V}^2_{ac}(k, p-k) \\
\bar{\Pi}^{(3)}_{bb^+}(k, \mathrm{i}\omega_n) &= (\beta/N)\sum_p [u^2_{p-k}(n^d - n^\beta_{p-k})/(\Omega^\beta_{p-k} - Y_d + \mathrm{i}\omega_n) \\
&\quad + v^2_{p-k}(1 + n^d + n^\alpha_{p-k})/(\Omega^\alpha_{p-k} + Y_d - \mathrm{i}\omega_n)]\bar{V}^2_{bd}(k, p-k) \\
\bar{\Pi}^{(3)}_{a^+b^+} &= \bar{\Pi}^{(3)}_{ba} = 0
\end{aligned} \tag{22}$$

with

$$n^{c,d} \equiv n(y_{c,d}) \qquad n^{\alpha,\beta}_k \equiv n(\omega^{\alpha,\beta}_k) \qquad \bar{V}_{ac,bd} = \frac{V_{ac,bd}}{J_0(S^2 + S - m^2)}.$$

4.2. Spectrum of low-energy excitations and critical fields

The expressions obtained make it possible to determine the spectrum renormalised in the first order of r_0^{-3}. Its reduced form is

$$\tilde{\Omega}^\mu_k = [(\tilde{\bar{A}}^\mu_k)^2 - (\tilde{\bar{B}}^\mu_k)^2]^{1/2} \pm \tilde{\bar{C}}^\mu_k \qquad (\mu = \alpha, \beta) \tag{23}$$

where

$$\begin{aligned}
&\tilde{\bar{A}}^\mu_k = \bar{A}_k + \Delta\bar{A}^{\mu(4)}_k + \Delta\bar{A}^{\mu(3)}_k \qquad \tilde{\bar{B}}^\mu_k = B_k + \Delta\bar{B}^{\mu(4)}_k + \Delta\bar{B}^{\mu(3)}_k \\
&\tilde{\bar{C}}^\mu_k = C_k + \Delta\bar{C}^{\mu(4)}_k + \Delta\bar{C}^{\mu(3)}_k \\
&\Delta\bar{A}^{\mu(e)}_k = -[\bar{\Pi}^{(e)}_{aa^+}(k, \pm\Omega^\mu_k) + \bar{\Pi}^{(e)}_{bb^+}(k, \mp\Omega^\mu_k)]/2\beta \\
&\Delta\bar{C}^{\mu(e)}_k = -[\bar{\Pi}^{(e)}_{aa^+}(k, \pm\Omega^\mu_k) - \bar{\Pi}^{(e)}_{bb^+}(k, \mp\Omega^\mu_k)]/2\beta \\
&\Delta\bar{B}^{\mu(e)}_k = -[\bar{\Pi}^{(e)}_{ba}(k, \mp\Omega^\mu_k) + \bar{\Pi}^{(e)}_{a^+b^+}(k, \mp\Omega^\mu_k)]/2\beta.
\end{aligned} \tag{24}$$

(Bars in $\bar{A}_k$, $\bar{B}_k$ and $\bar{C}_k$ mean reduction by the value of $J_0(S^2 + S - m^2)$; the upper signs correspond to $\mu = \alpha$.)

The explicit expressions for the spectrum will be written and analysed for the cases of $m = S$ and $m \neq S$ separately.

(i) The ordered phase with $m = S$. In this case

$$\begin{aligned}
\Delta\omega^\alpha_k &= \Delta A^{(4)}_k + \Delta C^{(4)}_k + \Delta A^{(3)}_k + \Delta C^{(3)}_k \\
&= (1/N)\sum_p n^\alpha_p\{-J_0(1 - \nu_k)(1 - \nu_p) + (2S - 1)J_0 D \\
&\quad \times (\nu_k + \nu_p)/[J_0 S(\nu_k + \nu_p) + 2D]\}
\end{aligned} \tag{25}$$

where $\Delta\omega^\alpha_k = \tilde{\omega}^\alpha_k - \omega^\alpha_k$.

Expression (25) coincides with the corresponding expression obtained in [8] for an easy-axis FM ($D < 0$), where for any values of anisotropy constant and field h at $T = 0$ a single phase, namely the saturated FM phase, is possible.

It should be noted also that at $D = 0$ the Goldstone theorem is fulfilled: $\omega_0^{\alpha} = 0$.

(ii) Partially ordered phases with $m \neq S$. In the vicinity of the multicritical point the renormalised spectrum at small k has the form

$$\tilde{\Omega}_k^{\alpha,\beta} = [\xi - \tilde{\xi}_A(T) + \tilde{Q}k^2]^{1/2} + \tilde{p}k \pm [H - \tilde{H}_A(T) + \tilde{V}\rho k^2] \tag{26}$$

with

$$\tilde{\xi}_A(T) = 1 - \delta\xi(0) - \delta\xi(T) \qquad \tilde{H}_A(T) = H_A - \delta H(0) - \delta H(T)$$

$$\delta\xi(0) = \left[-\frac{3}{2} + \frac{1}{2N}\sum_p \frac{3 + \nu_p f + \nu_p^2 F}{(1-\nu_p)^{1/2}}\right] - \left[\frac{1}{8N}\sum_p v_p^2(1+\nu_p)^2\left(\frac{(1+mF)(f+3mF)}{(1-\nu_p)^{1/2}+4+mF} + \frac{(1-mF)(f-3mF)}{(1-\nu_p)^{1/2}+4-mF}\right)\right] \qquad (f \equiv 1-2F) \tag{27}$$

$$\delta H(0) = \left[\frac{mF}{8}\left(-6 + \frac{1}{N}\sum_p \frac{6-\nu_p-\nu_p^2}{(1-\nu_p)^{1/2}}\right)\right] - \left[\frac{1}{8N}\sum_p v_p^2(1+\nu_p)^2 \times \left(\frac{(1+mF)(f+3mF)}{(1-\nu_p)^{1/2}+4+mF} - \frac{(1-mF)(f-3mF)}{(1-\nu_p)^{1/2}+4-mF}\right)\right]$$

$$\delta\xi(T) = \theta^2\rho^{-3/2}\zeta(2)[2 - N_+/16]/\pi^2$$

$$\delta H(T) = \theta^2\rho^{-3/2}\zeta(2)[mF - N_-/16]/\pi^2$$

$$N_\pm = \frac{(1+mF)(f+3mF)}{4+mF} \pm \frac{(1-mF)(f-3mF)}{4-mF}.$$

In (26) and (27)

$$v_p^2 = \frac{1}{2}\left(\frac{1-\nu_p/2}{(1-\nu_p)^{1/2}} - 1\right) \qquad \theta \equiv k_B T/J_0(S^2+S-m^2)$$

F is determined by (9) and $\zeta(p) = \Sigma_n n^{-p}$. The terms in the first large square brackets in expressions for $\delta\xi(0)$ and $\delta H(0)$ as well as the first terms in the brackets in expressions for $\delta\xi(T)$ and $\delta H(T)$ are given by $\mathcal{H}_4$ and others by $\mathcal{H}_3$. Expressions for $\tilde{P}$, $\tilde{Q}$ and $\tilde{V}$ are more cumbersome, and therefore they are not written out here. While calculating anharmonic corrections, we used the values $\xi = \xi_A^{(0)} = 1$.

The position of the multicritical points with the account of anharmonic effects is determined by the renormalised values of relative anisotropy and magnetic field $\tilde{\xi}_A(0)$ and $\tilde{H}_A(0)$ from (27). A numerical estimate using the values of integrals over the Brillouin zone for the simple cubic lattice

$$\frac{1}{N}\sum_k \frac{1}{(1-\nu_k)^{1/2}} \simeq 1.114 \qquad \frac{1}{N}\sum_k \frac{\nu_k}{(1-\nu_k)^{1/2}} \simeq 0.139$$

$$\frac{1}{N}\sum_k \frac{\nu_k^2}{(1-\nu_k)^{1/2}} \simeq 0.23$$

gives

$$\begin{aligned} &\delta\xi^{(4)}(0) \simeq 0.238 + 0.045F \qquad &&\delta\xi^{(3)}(0) \simeq -0.007N_+ \\ &\delta H^{(4)}(0) \simeq 0.04mF \qquad &&\delta H^{(3)}(0) \simeq -0.007N_-. \end{aligned} \tag{28}$$

Since the m- and S-dependent quantities included in $N_\pm$ vary over the range of $mF \in (0, 1/\sqrt{6})$, $f \in (0, 1)$ and correspondingly quantities $N_\pm$ vary over the range of $N_+ \in (0\text{–}0.5)$, $N_- \in (0\text{–}0.6)$, one can draw the following conclusion about anharmonic renormalisation of the position of the multicritical point. Renormalisation of relative anisotropy at the MP $\Delta\xi(0)$ is determined mainly by the four-operator anharmonisms; it is rather big and depends weakly on m and S on a background of the big contribution independent of the phase number and the spin value. On the contrary, in the critical field renormalisation at the MP $\Delta H(0)$ the contributions from $\mathcal{H}_4$ and $\mathcal{H}_3$ are comparable and the resulting value increases with the growth of m and decreases with the growth of S.

As to the temperature dependences, the critical field in the vicinity of the MP depends linearly on T:

$$\Delta H_{c1}^{(m)}(T) = \Delta H_{c2}^{(m)}(T) = \theta[\rho^{-3/2}\zeta(2)(2 - N_+/16)/\pi^2]^{1/2} \tag{29}$$

where

$$\Delta H_{c1}^{(m)}(T) \equiv \tilde{H}_{c1}^{(m)}(T) - \tilde{H}_{c1}^{(m)}(0) \qquad \Delta H_{c2}^{(m)}(T) \equiv \tilde{H}_{c2}^{(m)}(0) - \tilde{H}_{c2}^{(m)}(T).$$

In the region not directly close to the MP the renormalised spectrum has the form

$$\begin{aligned} &\tilde{\Omega}^{\alpha,\beta} = \tilde{\Delta}^{\alpha,\beta} + (U^{\alpha,\beta} + \Delta U^{\alpha,\beta})k^2 \\ &\tilde{\Delta}^{\mu} = \Delta^{\mu} + \delta\Delta^{\mu}(0) + \delta\Delta^{\mu}(T) \qquad (\mu = \alpha, \beta) \end{aligned} \tag{30}$$

with

$$\begin{aligned} \delta\Delta^{\alpha,\beta}(0) = \frac{1}{N}\sum_p \{&(\xi/2\Delta_1)[(3 + 2\nu_p)v_p^2 + 3\nu_p u_p v_p + f(\nu_p v_p^2 - u_p v_p)] \\ &\pm mF(3 + \nu_p)v_p^2\} - \frac{1}{N}\sum_p \left([v_p^2(1 + \nu_p)^2/4] \right. \\ &\left. \times \frac{v_0^2(1 \mp mF)(f \mp 3mF)}{\varepsilon_p + \varepsilon_0 - 4\xi \mp mF} + \frac{u_0^2(1 \pm mF)(f \pm 3mF)}{\varepsilon_p - \varepsilon_0 + 4\xi \pm mF} \right) \end{aligned} \tag{31}$$

and

$$\begin{aligned} \delta\Delta^{\alpha}(T) = \theta^{3/2}\Delta_1^{-2}[\Gamma(3/2)/4\pi^2][&(U^{\alpha})^{-3/2}R_1(\xi, m) \\ &\times Z_{3/2}(\Delta^{\alpha}/\theta) + (U^{\beta})^{-3/2}R_2(\xi, m)Z_{3/2}(\Delta^{\beta}/\theta)] \\ \delta\Delta^{\beta}(T) = \theta^{3/2}\Delta_1^{-2}[\Gamma(3/2)/4\pi^2][&(U^{\beta})^{-3/2}R_1(\xi, -m) \\ &\times Z_{3/2}(\Delta^{\beta}/\theta) + (U^{\alpha})^{-3/2}R_2(\xi, m)Z_{3/2}(\Delta^{\alpha}/\theta)] \end{aligned} \tag{32}$$

$$Z_p(x) \equiv \sum_{n=1}^{\infty} n^{-p}\, \mathrm{e}^{-nx}.$$

The values $R_1(\xi, m)$ and $R_2(\xi, m)$ are equal to

$$R_1(\xi, m) = \{\xi^2 + f\Delta_1^2 + 2mF\Delta_1(\xi - 1/2)\} - \{[2\xi(\xi - 1) + 1/4]N_+(\xi) + 2(\xi - 1/2)\Delta_1 N_-(\xi)\}/4 \tag{33}$$

$$R_2(\xi, m) = \{\xi^2 + F\Delta_1^2\} + [M_+(\xi)/16]$$

with

$$N_\pm(\xi) = \frac{(1 + mF)(f + 3mF)}{4\xi + mF - 2\Delta_1} \pm \frac{(1 - mF)(f - 3mF)}{4\xi - mF + 2\Delta_1}$$

$$M_+(\xi) = \frac{(1 + mF)(f + 3mF)}{4\xi + mF} + \frac{(1 - mF)(f - 3mF)}{4\xi - mF}.$$

In (30)–(33) $H_{c1}^{(m)}$ and $H_{c2}^{(m)}$ are the dimensionless critical fields in the zeroth approximation determined by (13); the values characterising the non-perturbative dispersion laws ε_p, u_p and v_p are determined by formulae (15) and (12); $\Delta_1 = \varepsilon_0$ and $\Delta^{\alpha,\beta}$ and $U^{\alpha,\beta}$ are determined by (17), F by (9). In the formulae for $\delta\Delta^{\alpha,\beta}(0)$ and $R_{1,2}(\xi, m)$ the expressions in the first braces are due to $\mathcal{H}_4$ and the others are due to $\mathcal{H}_3$. Values $N_\pm(\xi)$ and $M_+(\xi)$, reflecting the explicit dependence of relative temperature corrections from m and S, at $\xi = 1$ pass to $N_\pm$ from equation (27).

The lines of renormalised spectrum (30) softening determine the renormalised critical fields outside the closest vicinity of MP

$$\tilde{H}_{c2}^{(m)}(T) = [H_{c2}^{(m)} - \delta\Delta^\alpha(0)] - \delta H_{c2}(T) \qquad \tilde{H}_{c1}^{(m)}(T) = [H_{c1}^{(m)} + \delta\Delta^\beta(0)] + \delta H_{c1}(T)$$

$$\delta H_{c2}(T) = \theta^{3/2}\Delta_1^{-2}(U^\alpha)^{-3/2}R_1(\xi, m)\zeta(3/2)\Gamma(3/2)/4\pi^2 \tag{34}$$

$$\delta H_{c1}(T) = \theta^{3/2}\Delta_1^{-2}(U^\beta)^{-3/2}R_1(\xi, -m)\zeta(3/2)\Gamma(3/2)/4\pi^2$$

where $\delta\Delta^{\alpha,\beta}(0)$ are determined from (31).

Analysing formulae (30)–(32) one can draw the following conclusions about the role of anharmonic effects at $T = 0$. (i) Owing to anharmonic interaction the collinear phase existence region broadens, i.e. the existence region of the phases with spontaneously broken symmetry narrows, which is quite natural from general considerations. (ii) The critical field renormalisation is non-symmetric: $\delta\Delta^\alpha(0) > \delta\Delta^\beta(0)$. (iii) The m and S dependence of anharmonic corrections at $T = 0$ is the following: at $D/J_0 = \text{const}$, $S = \text{const}$ they decrease with increase of m; at $D/J_0 = \text{const}$, $m = \text{const}$ they increase with the growth of S. Thus, at fixed S anharmonic corrections at $T = 0$ are maximum for the non-magnetic phase with $m = 0$ and minimum (zero) for the saturated ferromagnetic phase with $m = S$. At first glance the S dependence of corrections might seem unusual; nevertheless it reflects the general tendency. Anharmonic effects are more pronounced the farther the phase is from the saturated ferromagnetic one, i.e. the bigger is the role of tensor components in the construction of the spin order at $T = 0$. At $m = \text{const}$ (and $m \neq S$) the growth of S is accompanied by the growth of $S - m$, a value that characterises the degree of deviation from the saturated FM phase.

For the fixed phase ($m, S = \text{const}$) anharmonic effects at $T = 0$ increase with the decrease of D/J_0 and become maximum in the vicinity of the MP; the field dependence of anharmonic corrections at $T = 0$ is absent.

4.3. Scattering amplitudes; excitation damping

Outside the close vicinity of MP it is convenient to pass to the concept of the quasi-particle scattering amplitudes with which the temperature parts of anharmonic corrections to various physical amplitudes are connected by simple linear relationships. For example, for the spectrum and free energy they have the form [4, 9]

$$\Delta\omega_k^{\mu}(T) = -(1/\beta N)\sum_{p\nu}\Gamma^{\mu\nu}(kp, pk)n(\omega_p^{\nu}) \tag{35}$$

$$\Delta F^{\text{int}}(T) = -(1/\beta N^2)\sum_{kp,\mu\nu}\Gamma^{\mu\nu}(kp, pk)n(\omega_k^{\mu})n(\omega_p^{\nu}). \tag{36}$$

In (35) and (36) $\Gamma^{\mu\nu}(kp, pk)$ are the forward scattering amplitudes corresponding to the normal scattering processes conserving the number of quasi-particles. The amplitudes corresponding to anomalous processes not conserving the number of quasi-particles in the first order in r_0^{-3} do not make contributions to anharmonic corrections, so in this order the interacting quasi-particle gas is described by the effective Hamiltonian conserving the quasi-particle number (for the phase with $m = S$ this is valid in any order in r_0^{-3} since $v_k = 0$). This is not valid in the vicinity of the MP where the scattering amplitudes corresponding to normal and anomalous processes are involved on equal grounds in all formulae since the non-perturbed frequency ε_k to which the correction from the normal process amplitude is added tends to zero at $k \to 0$.

In the first order in r_0^{-3} the scattering amplitudes $\Gamma^{\mu\nu}(kp, pk)$ are determined by the plots presented in figure 5. The explicit expressions for the forward scattering amplitudes have the form (for relative amplitudes $\bar{\Gamma} \equiv (\Gamma/J_0(S^2 + S - m^2))$:

$$\begin{aligned}
(1/\beta)\bar{\Gamma}^{\alpha\alpha(4)}(kp, pk) = &-\{(\nu_k + \nu_p)[2(u_k^2u_p^2 + v_k^2v_p^2 + u_kv_ku_pv_p) \\
&+ u_k^2v_p^2 + v_k^2u_p^2] + r[(\nu_k + 3\nu_p)u_pv_p(u_k^2 + v_k^2) \\
&+ (\nu_p + 3\nu_k)u_kv_k(u_p^2 + v_p^2)] + mF[2(\nu_k + \nu_p)(u_k^2u_p^2 - v_k^2v_p^2) \\
&+ (\nu_k - \nu_p)(u_k^2v_p^2 - u_p^2v_k^2)] - 2F(1 + \nu_{k-p})(u_k^2u_p^2 + v_k^2v_p^2) \\
&+ 2F(v_k^2u_p^2 + u_k^2v_p^2) + 4\nu_{k-p}Fu_kv_ku_pv_p\}/2 \\
(1/\beta)\bar{\Gamma}^{\alpha\beta(4)}(kp, pk) = &-\{(\nu_k + \nu_p)[2(u_k^2v_p^2 + v_k^2u_p^2 + u_kv_ku_pv_p) \\
&+ v_k^2v_p^2 + u_k^2u_p^2] + r[(\nu_k + 3\nu_p)u_pv_p(u_k^2 + v_k^2) + (\nu_p + 3\nu_k) \\
&\times u_kv_k(u_p^2 + v_p^2)] + mF[2(\nu_k + \nu_p)(u_k^2v_p^2 - v_k^2u_p^2) \\
&+ (\nu_k - \nu_p)(u_k^2u_p^2 - v_k^2v_p^2)] - 2F(1 + \nu_{k-p})(u_k^2v_p^2 + v_k^2u_p^2) \\
&+ 2F(u_k^2u_p^2 + v_k^2v_p^2) + 4F\nu_{k-p}u_kv_ku_pv_p\}/2
\end{aligned} \tag{37}$$

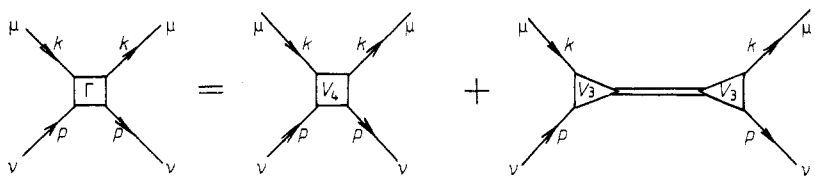

Figure 5. Diagrams for the forward scattering amplitudes $\Gamma^{\mu\nu}(kp, pk)$ of the zeroth order in r_0^{-3} ($\mu, \nu = \alpha, \beta$). V_3 and V_4 are the trial interactions determined by Hamiltonians $\mathcal{H}_3$ and $\mathcal{H}_4$ after performing u–v transformation. Other notation is the same as in figure 4.

and

$$\frac{1}{\beta}\bar{\Gamma}^{\alpha\alpha(3)}(kp,pk) = -\frac{\bar{V}^2_{ac}(k,p)u^2_k u^2_p}{\Omega^\alpha_k + \Omega^\alpha_p - Y_c} + \frac{\bar{V}^2_{bd}(k,p)v^2_k v^2_p}{\Omega^\alpha_k + \Omega^\alpha_p + Y_d}$$
$$\frac{1}{\beta}\bar{\Gamma}^{\alpha\beta(3)}(kp,pk) = -\frac{\bar{V}^2_{ac}(k,p)u^2_k v^2_p}{\Omega^\alpha_k - \Omega^\beta_p - Y_c} + \frac{\bar{V}^2_{bd}(k,p)v^2_k u^2_p}{\Omega^\alpha_k - \Omega^\beta_p + Y_d}. \tag{38}$$

Expressions (37) correspond to the first diagram in figure 5, expressions (38) to the second one; upper index '3' or '4' indicates contribution from $\mathcal{H}_3$ or $\mathcal{H}_4$, respectively. Total amplitudes are

$$\bar{\Gamma}^{\mu\nu}(kp,pk) = \bar{\Gamma}^{\mu\nu(4)}(kp,pk) + \bar{\Gamma}^{\mu\nu(3)}(kp,pk).$$

(Expressions for $\bar{\Gamma}^{\beta\beta(3)}(kp,pk)$ and $\bar{\Gamma}^{\beta\beta(4)}(kp,pk)$ can be obtained from the expressions for $\bar{\Gamma}^{\alpha\alpha(3)}(kp,pk)$ and $\bar{\Gamma}^{\alpha\alpha(4)}(kp,pk)$, respectively, by replacement $\alpha \rightleftarrows \beta, c \rightleftarrows d$.) In the limit of small k and p determining the values of integrals by wavevectors at low T occurring in the calculation of various physical characteristics, the amplitudes are equal to

$$-\frac{1}{\beta}\bar{\Gamma}^{\mu\mu}(00,00) = \frac{R_1(\xi,\pm m)}{\Delta_1^2} \qquad -\frac{1}{\beta}\bar{\Gamma}^{\alpha\beta}(00,00) = \frac{R_2(\xi,m)}{\Delta_1^2}. \tag{39}$$

(The '+' sign corresponds to $\mu = \alpha$, the '−' sign to $\mu = \beta$.) Let us analyse the dependence of long-wave amplitudes on m, S, D/J_0 and h/J_0. It can be easily seen that within each phase the absolute values of amplitudes (39) decrease with the growth of D/J_0 and do not depend on h/J_0. At $D/J_0 = \text{const}$, $S = \text{const}$ the absolute magnitudes of long-wave amplitudes decrease as the phase number m increases; at $D/J_0 = \text{const}$, $m = \text{const}$ they increase as S does. This is valid for the phases with $m \neq S$ when the contribution from four-particle processes (the first terms in expression (33) for $R_{1,2}(\xi, m)$) is predominant. For $m = S$, when at small k and p the contribution from three-particle processes is predominant, the dependences are the opposite: increase with growth of D/J_0 at $S = \text{const}$ and decrease with growth of S at $D/J_0 = \text{const}$ (explicit form of amplitudes for $m = S$ will be given below in (43)).

According to (37) and (38) for any phase m at small k and p there is quasi-particle repulsion; then at some k and p the amplitudes pass through zero; and then quasi-particle attraction appears, which enhances as k and p grow. This attraction remains finite until conditions

$$Y_c > \max_p(\Omega^\alpha_k + \Omega^\alpha_p) \qquad Y_d > \max_p(\Omega^\beta_k + \Omega^\beta_p) \tag{40}$$

are fulfilled. These are the conditions of stability against merging of two quasi-particles into one excitation of the c or d type. If k achieves such values k^α_0 and k^β_0, starting from which for some p inverse inequalities are fulfilled then, at $k > k^\mu_0$ the Ω^μ_k spectrum becomes unstable against these merging processes. The respective damping is determined by formula

$$\gamma^{(3)}_{\mu k} = \frac{1}{\beta}\sum_p [n^\mu_p \operatorname{Im}\Gamma^{\mu\mu}(kp,pk) + n^\nu_p \operatorname{Im}\Gamma^{\mu\nu}(kp,pk)] \tag{41}$$

and is equal to

$$\gamma_{\mu k}^{(3)} \sim r_0^{-3}[k^2 - (k_0^\mu)^2]^{1/2}\theta(k - k_0^\mu)[Av_k^2 + Bn(Y_l - \Omega_k^\mu)]. \tag{42}$$

(In (42), $n(x)$ is the Bose factor,

$$\theta(x) = \begin{cases} 1 & x > 0 \\ 0 & x < 0 \end{cases}$$

in Y_l index $l = c$ for $\mu = \alpha$ and $l = d$ for $\mu = \beta$.) One can easily prove that for all partially ordered phases with $m \neq S$ at any k and p conditions (40) are fulfilled at relative anisotropy values $\xi > 1$ corresponding to the existence regions of these phases and damping (41) is absent.

For the phase with $m = S$ for which there is only c mode and the explicit form of the scattering amplitude at arbitrary k and p is

$$\Gamma^{\alpha\alpha}(kp, pk)/\beta J_0 = (1 - \nu_k)(1 - \nu_p) - 2(D/J_0)(2S - 1)\frac{\nu_k + \nu_p}{S(\nu_k + \nu_p) + 2D/J_0} \tag{43}$$

the same situation is realised at $D > SJ_0$ (which also corresponds to $\xi > 1$). At $D < SJ_0$ (i.e. $\xi < 1$ coming into the existence region of this phase the magnitude of k_0^α is determined by the equality

$$\nu_{k_0^\alpha} = 1 - 2D/SJ_0 \tag{44}$$

where ν_k is determined by (11). Thus with decrease of D the quasi-particle damping (42) appears (but only at $T \neq 0$ since $v_k^2 = 0$ for $m = S$) first at $D = SJ_0$ only at the Brillouin zone boundary, and then the region of wavevectors with non-zero damping increases and at $D = 0$ coincides with all the Brillouin zone; the magnitude of damping decreases with decreasing D.

A similar but mirror-symmetric picture is also typical of the phase with $m = S$ at $D < 0$ when the saturated ferromagnetic phase is the only possible phase at $h = h_z$. In this case amplitude (43) describes quasi-particle attraction at small k and p with the following passing through zero and repulsion at large k and p. The condition of stability against merging processes has the form $Y_c \leqslant \min_p(\Omega_k^\alpha + \Omega_p^\alpha)$, and is fulfilled at $|D| > SJ_0$. At $|D| < SJ_0$ instability of the long-wave part of the spectrum appears at $k < k_0^\alpha$. The magnitude of k_0^α is determined by equality $\nu_{k_0^\alpha} = 2|D|/SJ_0 - 1$, and damping due to these processes is determined by (42) on replacement $k - k_0^\mu \to -(k - k_0^\mu)$ and change of the sign of the arguments of the function $n(x)$.

It should also be noted that the ordered phase with $m = S$ is, on the one hand, the limiting case of phases in the cascade of field- or anisotropy-induced phase transitions and, on the other hand, it is the only phase in the cascade whose low-temperature dynamics can be described by means of the standard Maleev–Dyson and Holstein–Primakoff transformations. This makes it possible to compare the results of this paper with known results and, on the whole, the results of the approach using the special transformation to the Bose operators for the algebra su(n) [6] with the results based on the Maleev–Dyson representation in the region where both approaches are applicable. Analysing the results obtained we can make the following statements: (i) fulfilment of all symmetry requirements, namely the Goldstone theorem and the Adler principle at

$D = 0$, and kinematic condition at $S = 1/2$†; (ii) qualitative coincidence of the described behaviour of scattering amplitude and excitation damping in the case of $D < 0$ with the exact results obtained for the easy-axis ferromagnet [10]; (iii) full quantitative coincidence of the results for the easy-axis case with those of [8, 11] obtained on the basis of the diagram technique for the Hubbard operators. Anharmonic effects in the case of the SA easy-plane symmetry have not been considered previously (with the exception of the limiting case $D/J_0 \ll 1$; see, for example, [12, 13]).

In addition to damping (41) due to the three-particle processes and corresponding to the first order in r_0^{-3} there is, as usual (e.g. [9]), damping due to the four-particle scattering processes and corresponding to the second order in r_0^{-3}. In the general case for an arbitrary phase m it is determined by the formula

$$\gamma_{\mu k}^{(4)} = (8\pi/\beta^2 n_k^\mu) \sum_{qp} [n_p^\mu n_q^\mu (1 + n_{q+p-k}^\mu) |\Gamma^{\mu\mu}(k, p+q-k; pq)|^2 + n_p^\mu n_q^\nu (1 + n_{q+p-k}^\nu) |\Gamma^{\mu\nu}(k, p+q-k; pq)|^2] \times \delta(\varepsilon_p + \varepsilon_q - \varepsilon_{q+p-k} - \varepsilon_k). \tag{45}$$

At small k explicit calculations give

$$\gamma_{\mu k}^{(4)} \sim r_0^{-6} \, \mathrm{e}^{-\Delta\mu/\theta} \, \Delta_1^{-4} \theta^2 R_1^2(\xi, \pm m). \tag{46}$$

4.4. Exciton-type modes

In the foregoing we considered the behaviour of two low-lying modes. It is interesting to note that due to three-bosonic-operator interaction dispersion also appears in two modes localised in the spin-wave approximation, namely in the c- and d-type modes. In fact, in the Born approximation their dimensionless frequencies have the form

$$\tilde{\Omega}_k^c = Y_c - \frac{1}{\beta} \bar{\Pi}_{cc^+}(k, Y_c) \qquad \tilde{\Omega}_k^d = Y_d - \frac{1}{\beta} \bar{\Pi}_{dd^+}(k, Y_d). \tag{47}$$

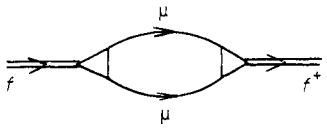

Figure 6. Diagrams of the first order in r_0^{-3} for polarisation operators of high-lying excitations Π_{cc^+} and Π_{dd^+}. Notation is the same as in figure 4.

The plots for polarisation operators $\bar{\Pi}_{cc^+}$ and $\bar{\Pi}_{dd^+}$ are shown in figure 6. The corresponding analytical expressions are

$$\bar{\Pi}_{cc^+}(k, \mathrm{i}\omega_n) = (\beta/N) \sum_p \bar{V}_{ac}^2(p, k-p) \left(\frac{u_p^2 u_{k-p}^2 (1 + n_p^\alpha + n_{k-p}^\alpha)}{\Omega_p^\alpha + \Omega_{k-p}^\alpha - \mathrm{i}\omega_n} + \frac{u_p^2 v_{k-p}^2 (n_p^\alpha - n_{k-p}^\beta)}{\Omega_{k-p}^\beta - \Omega_p^\alpha + \mathrm{i}\omega_n} + \frac{v_k^2 v_{k-p}^2 (1 + n_p^\beta + n_{k-p}^\beta)}{\Omega_p^\beta + \Omega_{k-p}^\beta + \mathrm{i}\omega_n} + \frac{v_p^2 u_{k-p}^2 (n_p^\beta - n_{k-p}^\alpha)}{\Omega_{k-p}^\alpha - \Omega_p^\beta - \mathrm{i}\omega_n} \right). \tag{48}$$

† For phases with $m \neq S$ the Adler principle cannot be fulfilled since these phases do not allow the limiting transition to $D = 0$, and the kinematic condition is provided by the fact that at $S = \frac{1}{2}$ the maximum critical field at $T = 0$, $h_{c2}^{(s)} = D(2S-1)$, is equal to zero. Thus only one phase with $m = S$ is realised and all effects associated with the existence of partially ordered phases disappear.

(Formula for $\bar{\Pi}_{dd^+}$ can be obtained from (48) by replacement $\Omega^\alpha \rightleftarrows \Omega^\beta$, $\bar{V}_{ac} \to \bar{V}_{bd}$.) From (47) and (48) one can easily get an explicit form of dispersion laws of the modes under consideration. In particular, at $T = 0$ we have

$$\tilde{\Omega}_k^c = Y_c - \frac{1}{N}\sum_p \left(\frac{u_p^2 u_{k-p}^2}{\Omega_p^\alpha + \Omega_{k-p}^\alpha - Y_c} + \frac{v_k^2 v_{k-p}^2}{\Omega_p^\beta + \Omega_{k-p}^\beta + Y_c}\right)\bar{V}_{ac}^2(p, k-p). \tag{49}$$

For the phases with $m \neq S$ these modes are non-damping at $T = 0$ for all values of wavevectors since, as was mentioned before, the inelastic three-particle processes (in this case the processes of decay of the c- and d-type excitations into two quasi-particle of α and β type) are only virtual at $\xi > 1$ corresponding to the existence regions of these phases. The same is valid for the phase with $m = S$ at $\xi > 1$, i.e. $D > SJ_0$. At $\xi < 1$ there is a critical value of wavevector k_0 determined by equality† $\nu_{k_0/2} = D/SJ_0$ such that at $k < k_0$ damping at $T = 0$ appears given by

$$\gamma_k^c \sim D^2(2S-1)(k_0^2 - k^2)^{1/2}\theta(k_0 - k). \tag{50}$$

It should be noted that with the decrease of D the region of wavevectors where there is damping (50) broadens. On the other hand, however, damping intensity decreases so that at whatever small D there is a region of the stability modes in the Brillouin zone angles.

4.5. Free energy

The above modes are of the exciton type. They are essential in the investigation of optical properties, ferromagnetic resonance and so on, but produce almost no effect on the system thermodynamics at low temperatures. The system thermodynamics is determined by the free energy form whose temperature part consists of two terms. The free energy of non-interacting quasi-particles (not trial ones but with the account of spectrum renormalisation at $T = 0$) is described by the standard expression

$$F(T) = \frac{1}{\beta}\sum_{k\mu} \ln(1 - \mathrm{e}^{-\beta\omega_k^\mu}).$$

The correction due to the dynamic quasi-particle interaction outside the closest vicinity of MP has the form (36). Explicit calculations give

$$\begin{aligned}\Delta F^{\mathrm{int}}(T) = J_0 S(S+1)\theta^3 \Delta_1^{-2}[\Gamma(3/2)/4\pi^2]^2[R_1(\xi, m) \\ \times (U^\alpha)^{-3} Z_{3/2}^2(\Delta^\alpha/\theta) + R_1(\xi, -m)(U^\beta)^{-3} Z_{3/2}^2(\Delta^\beta/\theta) \\ + R_2(\xi, m)(U^\alpha U^\beta)^{-3/2} Z_{3/2}(\Delta^\alpha/\theta) Z_{3/2}(\Delta^\beta/\theta)].\end{aligned} \tag{51}$$

Knowing explicit expressions for the free energy, one can determine the behaviour of any thermodynamic functions on the basis of thermodynamic relationships.

5. Antiferromagnetic exchange case

The Hamiltonian has the same form as in the ferromagnetic exchange case but $J_{ij} < 0$. Respectively in the Gaussian approximation the spectrum of two low-energy modes with

† This equality is obtained from the condition of stability against decay $Y_c(k) \geqslant \max_p(\Omega_p^\alpha + \Omega_{k-p}^\beta)$, i.e. condition $D \geqslant SJ_0\nu_{k/2}$.

dispersion for collinear phases with number m (the phases with flipped sublattices and sublattice magnetisation $\langle S^z\rangle_f = m$) has the form

$$\Omega_k^{\alpha,\beta} = \varepsilon_k \pm [H - m(2\xi + F - F\nu_k/2]$$
$$\varepsilon_k = \{[\xi + (r+1)\nu_k/2][\xi - (r-1)\nu_k/2]\}^{1/2} \tag{52}$$

where

$$\xi = D/[|J_0|(S^2 + S - m^2)] \qquad H = h/[|J_0|(S^2 + S - m^2)]$$

$$\Omega_k^{\mu} = \frac{\omega_k^{\mu}}{|J_0|(S^2 + S - m^2)}$$

other notations being the same as in the FM case. The value of r is determined by formula (14). The frequencies of the c- and d-type modes are

$$Y_c = 4\xi(1 - m) + 2(H - mF) \qquad Y_d = 4\xi(1 + m) - 2(H - mF). \tag{52a}$$

Functions of the u–v transforms are

$$u_k^2 = \frac{1}{2}\left(\frac{\xi + \nu_k/2}{\varepsilon_k} + 1\right) \qquad v_k^2 = \frac{1}{2}\left(\frac{\xi + \nu_k/2}{\varepsilon_k} - 1\right). \tag{53}$$

Frequencies Ω_k^{μ} decrease with the growth of wavevector magnitude. Their softening lines at the wavevectors corresponding to the Brillouin zone boundary ($\nu_k = -1$) determine the lines of the second-order phase transition

$$H_{c1,c2}^{(m)} = \Delta_2 \pm \Delta_1$$
$$\Delta_2 = m(2\xi + 3F/2) \qquad \Delta_1 = \varepsilon_0 = \begin{cases} [\xi(\xi - 1)]^{1/2} & m \neq S \\ \xi - 1/2 & m = S \end{cases} \tag{54}$$

to non-collinear antiferromagnetic phases whose spin-order structure differs from the order structure in non-collinear FM phase discussed before only in that, instead of perpendicular magnetisation component (easy-plane magnetisation), they have perpendicular magnetisation component of the sublattice. Indeed, in the non-collinear FM phases, the change of the magnetic field from $H_{c1}^{(m)}$ to $H_{c2}^{(m+1)}$ at $D/J_0 = \text{const}$ rotates the magnetisation vector from the direction parallel to the external field to a direction forming the maximum angle with the z axis at some point within the phase region and then brings this vector back to the z axis. And in the AFM case, rotation from the flipping sublattices with magnetisation vectors parallel to the magnetic field to the maximum angle between these vectors at some point within the interval $(H_{c1}^{(m)}, H_{c2}^{(m+1)})$ and then reverse rotation to the flipping sublattices occurs. The magnitude of the maximum critical field at fixed D/J_0 is $h_{c2}^{(s)} = h - D(2S - 1) - 2|J_0|$. The coordinates of multicritical points, which are determined by equality $H_{c1}^{(m)} = H_{c2}^{(m)}$, are

$$\xi_A = 1 \qquad H_A = m(2 + 3F/2). \tag{55}$$

At these points the dispersion law of soft modes (two degenerate modes at the Brillouin zone boundary) is linear as opposed to all other points of lines $H_{ci}^{(m)}(\xi)$ where there is one soft mode with quadratic dispersion law. On the whole, the phase diagram structure is the same as in the case of the FM exchange.

Prior to investigating the anharmonicity effects, let us give an explicit form of the low-lying excitation dispersion law at wavevector values in the vicinity of the Brillouin

zone angle (i.e. in the low-energy limit) for $m \neq S$:

$$\Omega_k^\mu = \Delta^\mu + W^\mu q^2 \qquad (\mu = \alpha, \beta, q = k_b - k, \nu_{k_b} = -1)$$
$$\Delta^\alpha = H - H_{c2}^{(m)} \qquad \Delta^\beta = H_{c1}^{(m)} - H \qquad W^{\alpha,\beta} = \rho\left(\frac{\xi}{2\Delta_1} \pm \frac{mF}{2}\right). \tag{56}$$

This expression is valid outside the multicritical points. For the phase with $m = S$ there is one mode

$$\Omega_k^\alpha = H - (2S-1)\xi + 1 + \nu_k. \tag{57}$$

Anharmonic corrections are described by the same formulae as in section 4 with replacement $J_0 \to -|J_0|$, i.e. by formulae (35), (41), (45), (23) and (24), where polarisation operator components and scattering amplitudes are determined by formulae (21)–(22) and (37)–(38) using u_k^2, v_k^2, $\Omega_k^{\alpha,\beta}$ and $Y_{c,d}$ given by equations (52)–(53). It should be noted that in expressions containing sums over wavevectors the main contribution to these sums is made by wavevector values close to the Brillouin zone boundary. For example, the temperature corrections for the Brillouin zone boundary frequencies $\Omega_{k_b}^\mu$ governing the temperature behaviour of critical fields are determined by the amplitudes $\Gamma^{\mu\nu}(k_b p_b, p_b k_b)$, and the temperature corrections for the Brillouin zone centre frequencies Ω_0^μ governing the temperature dependence of the uniform ferromagnetic resonance and the Raman light scattering frequencies are determined by the amplitudes $\Gamma^{\mu\nu}(0p_b, p_b 0)$.

Analysing the behaviour of scattering amplitudes at various k and p one can see that qualitatively it is the mirror reflection of the FM case, namely, at small magnitudes of k and p the quasi-particles of identical type are attracted and at large magnitudes of k and p they are repulsed. Thus the amplitudes we are interested in are†

$$\frac{1}{\beta}\bar{\Gamma}^{\mu\mu}(k_b k_b, k_b k_b) = -\frac{Q_1(\xi, \pm m)}{\xi(\xi-1)} \qquad \frac{1}{\beta}\bar{\Gamma}^{\alpha\beta}(k_b k_b, k_b k_b) = -\frac{Q_2(\xi, m)}{\xi(\xi-1)}$$
$$Q_1(\xi, m) = [\xi^2 + (1+2F)\Delta_1^2 + 2mF\Delta_1(\xi - \tfrac{1}{2})] + \tfrac{1}{4}\{[2\xi(\xi-1) + \tfrac{1}{4}]N_+(\xi) + 2(\xi - \tfrac{1}{2})\Delta_1 N_-(\xi)\} \tag{58a}$$
$$Q_2(\xi, m) = (\xi^2 - F\Delta_1^2) - [M_+(\xi)/16]$$

and

$$\frac{1}{\beta}\bar{\Gamma}^{\mu\mu}(0k_b, k_b 0) = -\frac{P_1(\xi, \pm m)}{[\xi(\zeta-1)]^{1/2}[\xi(\zeta+1)]^{1/2}}$$
$$\frac{1}{\beta}\bar{\Gamma}^{\alpha\beta}(0k_b, k_b 0) = -\frac{P_2(\xi, m)}{\xi(\xi^2-1)^{1/2}}$$
$$P_1(\xi, m) = \xi\{F[\xi - (\xi^2-1)^{1/2}] - 1 + mF[(\xi+1)^{1/2}(\xi - \tfrac{1}{2}) - (\xi-1)^{1/2}(\xi + \tfrac{1}{2})]/\xi^{1/2}\}/2 \tag{58b}$$
$$P_2(\xi, m) = \xi\{F[\xi + (\xi^2-1)^{1/2}] - 1 + mF[(\xi+1)^{1/2}(\xi - \tfrac{1}{2}) + (\xi-1)^{1/2}(\xi + \tfrac{1}{2})]/\xi^{1/2}\}/2.$$

† It is interesting to note that the values of amplitudes $\Gamma^{\mu\nu}(0p_b, p_b 0)$ are determined exclusively by four-particle scattering processes since the contribution from three-particle processes proportional to $\nu_k + \nu_p$ is equal to zero at $k = 0, p = p_b$.

In (58*a*) and (58*b*)

$$\tilde{\Gamma}^{\mu\nu} \equiv \Gamma^{\mu\nu}/|J_0|(S^2 - m^2 + S)$$

and formulae for $N_{\pm}(\xi)$ and $M_{+}(\xi)$ coincide with formulae (33). The '+' sign in $Q_1(\xi, \pm m)$ and $P_1(\xi, \pm m)$ corresponds to $\mu = \alpha$, the '−' sign to $\mu = \beta$. The values $Q_1(\xi, m)$, $Q_2(\xi, m)$ and $P_2(\xi, m)$ are positive at any ξ from the existence region of phases with $m \neq S$; $P_1(\xi, m)$ is negative at sufficiently large ξ corresponding to the region far from the multicritical points. The difference in the scattering amplitude signs for identical quasi-particles in cases (58*a*) and (58*b*) results in different signs of temperature corrections to the frequencies at the Brillouin zone boundary and centre

$$\begin{aligned}
\Delta\Omega^{\alpha}_{k_b}(T) &= \theta^{3/2}\Delta_1^{-2}[\Gamma(3/2)/4\pi^2][(W^{\alpha})^{-3/2}Q_1(\xi, m)Z_{3/2}(\Delta^{\alpha}/\theta) \\
&\quad + (W^{\beta})^{-3/2}Q_2(\xi, m)Z_{3/2}(\Delta^{\beta}/\theta)] \\
\Delta\Omega^{\beta}_{k_b}(T) &= \theta^{3/2}\Delta_1^{-2}[\Gamma(3/2)/4\pi^2][(W^{\beta})^{-3/2}Q_1(\xi, m)Z_{3/2}(\Delta^{\beta}/\theta) \\
&\quad + (W^{\alpha})^{-3/2}Q_2(\xi, m)Z_{3/2}(\Delta^{\alpha}/\theta)] \\
\Delta\Omega^{\alpha}_{0}(T) &= \theta^{3/2}[\xi(\xi^2 - 1)^{1/2}]^{-1}[\Gamma(3/2)/4\pi^2][(W^{\alpha})^{-3/2}P_1(\xi, m)Z_{3/2}(\Delta^{\alpha}/\theta) \\
&\quad + (W^{\beta})^{-3/2}P_2(\xi, m)Z_{3/2}(\Delta^{\beta}/\theta)] \\
\Delta\Omega^{\beta}_{0}(T) &= \theta^{3/2}[\xi(\xi^2 - 1)^{1/2}]^{-1}[\Gamma(3/2)/4\pi^2][(W^{\beta})^{-3/2}P_1(\xi, -m) \\
&\quad \times Z_{3/2}(\Delta^{\beta}/\theta) + (W^{\alpha})^{-3/2}P_2(\xi, m)Z_{3/2}(\Delta^{\alpha}/\theta)]
\end{aligned} \tag{59}$$

where

$$\Delta\Omega^{\mu}_{k}(T) \equiv \tilde{\Omega}^{\mu}_{k}(T) - \tilde{\Omega}^{\mu}_{k}(0).$$

As a result, a qualitatively different frequency–temperature behaviour at the Brillouin zone boundary and centre is observed. Namely, at $k = k_b$ the frequencies of both excitation modes increase with T. At $k = 0$ the frequency of the α mode decreases with T for values of H close to $H_{c2}^{(m)}$ and increase for values of H close to $H_{c1}^{(m)}$; the behaviour of the β-mode frequency is quite the opposite. Respectively, critical field $\tilde{H}_{c1}^{(m)}(T)$ grows with T and critical field $\tilde{H}_{c2}^{(m)}(T)$ decreases with T. The behaviour of the Raman light scattering frequencies and uniform ferromagnetic resonance frequencies coincide with the behaviour of $\tilde{\Omega}_0^{\alpha,\beta}(T)$.

For the phase with $m = S$ the form of the scattering amplitude $\Gamma^{\alpha\alpha}(kp, pk)$ at arbitrary k and p is determined by equation (43). Hence $\Gamma^{\alpha\alpha}(0p_b, p_b0) = 0$. The form of the anharmonic correction $\Delta\Omega_0^{\alpha}(T)$ is determined by amplitude $\Gamma^{\alpha\alpha}(0p, p0)$ at small magnitudes of wavevector $q = p_b - p$. It is given by the expression

$$\Delta\omega_0^{\alpha}(T) = -|J_0|(2S - 1)\theta^{5/2}\rho Z_{5/2}\left(\frac{H - H_{c2}^{(S)}}{\theta}\right)\frac{\Gamma(5/2)}{4\pi^2}. \tag{60}$$

The behaviour of the anharmonic correction $\Delta\omega^{\alpha}_{k_b}$ is similar to the behaviour for phases with arbitrary m:

$$\Delta\omega^{\alpha}_{k_b}(T) = |J_0|\rho\left(4 + \frac{2\xi(2S - 1)}{1 + \xi}\right)\theta^{3/2}Z_{3/2}\left(\frac{H - H_{c2}^{(S)}}{\theta}\right)\frac{\Gamma(3/2)}{4\pi^2}. \tag{61}$$

Consideration of collective excitation damping due to inelastic processes (i.e. damping appearing in the first order in r_0^{-3}) leads to the following result. As in the FM exchange

case, the excitation damping $\gamma_{\mu k}^{(3)}$ is absent at any k for phases with $m \neq S$, which is due to the limitation of the relative anisotropy constant characterising the region within which these phases exist ($\xi \geqslant 1$). For the phase with $m = S$, damping $\gamma_{\alpha k}^{(3)}$ is absent at $\xi \equiv D/S|J_0| > 1$. At $\xi < 1$ it is

$$\gamma_{\alpha k}^{(3)} \sim r_0^{-3}(k_0^2 - k^2)^{1/2}\theta(k_0 - k)n(Y_c - \Omega_k^{\alpha})$$
$$\nu_{k_0} = 2D/S|J_0| - 1. \tag{62}$$

The difference from the ferromagnetic case consists of the following. When the anisotropy attains its critical value, the damping appears starting from the Brillouin zone centre but not from its boundary. When D decreases the region of wavevectors where damping (62) is observed increases and at $D = 0$ coincides with the entire Brillouin zone. The damping magnitude, however, decreases and tends to zero at $D \to 0$. Damping (62) determines the linewidths of uniform ferromagnetic resonance and Raman light scattering at low T. The short-wave portion of the spectrum determining the critical behaviour of the system remains stable against inelastic processes.

Damping of α and β quasi-particles due to four-particle processes has qualitatively the same form as in the FM exchange case (the value $Q_1(\xi, m)$ appears in formula (46) instead of $R_1(\xi, m)$). The same is also valid for the frequency shift and damping of high-lying c- and d-type excitations. The general form of equations (47)–(50) is preserved, only one has to substitute Ω_k^{μ}, $Y_{c,d}$, u_k^2 and v_k^2 from (52), (52a) and (53). As in the FM case these modes prove to be stable at $T = 0$ in the entire existence region of the phases with $m \neq S$ and in the region of large relative magnitudes of anisotropy $D/S|J_0| > 1$ for the phase with $m = S$. When the anisotropy constant becomes critical ($D = S|J_0|$), the excitation damping originates at the Brillouin zone centre. As D continues to diminish, this damping occurs in the ever-increasing part of the Brillouin zone. It has the form (50) with replacement $D/J_0 \to D/|J_0|$.

6. Some peculiarities of the critical behaviour at quantum field- and anisotropy-induced reorientational phase transitions

Let us discuss now some peculiarities of the critical behaviour at the quantum reorientational phase transitions (RPT) (transitions at $T = 0$) induced by magnetic field and single-ion anisotropy†. Naturally, perturbation theory is not valid for a quantitative description of the second-order phase transition but some qualitative conclusion as to the critical behaviour of the system may be drawn by using the spin-wave approximation.

First of all one should state that when the Hamiltonian includes external fields of two types—magnetic $h^{\alpha}S^{\alpha}$ and quadrupolar (SA fields) $D^{\pm m}(O_2^m \pm O_2^{-m})$—the system response to external actions is determined by the isothermal static susceptibility tensor (8×8) including the magnetic susceptibility tensor components $\chi_M^{\alpha\beta} = -\partial^2 F/\partial h^{\alpha}\partial h^{\beta}$, quadrupolar susceptibility tensor components $\chi_Q^{mn} = -\partial^2 F/\partial D^m \partial D^n$ and mixed susceptibilities $\chi^{\alpha n} = -\partial^2 F/\partial h^{\alpha}\partial D^n$. When studying $D \equiv D^0$ anisotropy-induced reorientational phase transition (RPT), of special interest is the behaviour of the quadrupolar susceptibility χ_Q^{00}, and in the event of $h \equiv h_z$ field-induced RPT the magnetic susceptibility

† As to the critical behaviour at a field- or anisotropy-induced RPT in the case of $T \neq 0$ it should be noted that at arbitrary small but finite T there is an arbitrary small but finite fluctuational region where the system's critical behaviour coincides with the behaviour at the temperature-induced phase transition (classical PT), which is described by the known scaling theories; see for example [14], and see too [15, 16].

χ_M^{zz} presents special interest, both characteristics playing the same part as the heat capacity $C = -T(\partial^2 F/\partial T^2)$ in the case of temperature-induced phase transitions. Let us consider their behaviour in the vicinity of the RPT points within the collinear and non-collinear phases.

6.1. *Collinear phases*

Magnetic susceptibility χ_M^{zz} is identically equal to zero since $\langle S^z \rangle = m$ is an exact equality at $T = 0$. The critical behaviour of the quadrupolar susceptibility χ_Q^{00} can be determined from the general form of the connection of the fluctuation Gaussian correction to the ground-state energy with the spectrum (see e.g. [4]):

$$\Delta E_0 = \tfrac{1}{2} \sum_{k\mu} (\omega_k^\mu - A_k^\mu) \tag{63}$$

(A_k^μ are the factors of the bilinear form $\mathcal{H}_2$) and the expressions for the spectrum (11) or (52). Preserving the most singular terms one obtains

$$\chi_Q^{00} \sim (\Delta\xi)^{(d-3)/2} \tag{64}$$

where $\Delta\xi = |\xi - \xi_A|$ is the distance to the multicritical point.

Thus we can conclude that, within the collinear phase, the critical behaviour is observed only at the multicritical points when the RPT is anisotropy-induced. And the border line dimension of space according to (63) is 3 ($d_c = 3$). At other points of lines $H_{cl}^{(m)}(\xi)$ the critical fluctuations are absent.

6.2. *Non-collinear phases*

Let us start from expression (63) for ΔE_0 and the expression for the Goldstone mode spectrum, which in the vicinity of lines $H_{cl}^{(m)}(\xi)$ but outside the close vicinity of the multicritical points has the form [1]

$$\omega_k = [Ak^2(Ak^2 + B\tau)]^{1/2} \tag{65}$$

where $\tau = \Delta h = |h_{cl}^{(m)} - h|$ in the case of the field-induced RPT and $\tau = \Delta D = |D_{cl}^{(m)} - D|$ in the case of the anisotropy-induced RPT. Respectively, for the fluctuation parts of the susceptibilities one obtains

$$\begin{aligned} \Delta\chi_Q^{00} &\sim \frac{1}{N}\sum_p [p/(Ap + B\Delta D)^{3/2}] \sim (\Delta D)^{d/2-1} \\ \Delta\chi_M^{zz} &\sim \frac{1}{N}\sum_p [p/(Ap + B\Delta h)^{3/2}] \sim (\Delta h)^{d/2-1} \end{aligned} \tag{66}$$

so that the border line dimension is equal to 2 ($d_c = 2$) for the field- or anisotropy-induced RPT points (except the multicritical points).

For the anisotropy-induced RPT at the multicritical points the critical mode is not the Goldstone but the optical one† whose spectrum has the form

$$\omega_k = (\alpha\Delta\xi + \beta k^2)^{1/2}. \tag{67}$$

† This was shown in [17] where the collinear phase behaviour in the vicinity of multicritical point $A^{(0)}$ at $S = 1$ was investigated. This behaviour, however, is universal for any multicritical point $A^{(m)}$.

Hence one obtains

$$\Delta\chi_{Q}^{00} \sim (\Delta\xi)^{(d-3)/2} \tag{68}$$

i.e. the same behaviour as that observed within the collinear phases. The field-induced RPT at the multicritical points is impossible since at $\xi = \xi_A$ and change in the field on both sides from the MP, the non-collinear phase is realised.

It should be noted that the critical behaviour of the system under consideration at the field- or anisotropy-induced RPT points except the multicritical points qualitatively coincide with the critical behaviour of the XY model at the RPT induced by a field perpendicular to the XY plane. (As was shown in [18–21] using the phenomenological approach, this behaviour is characterised by $d_c = 2$ and by the coincidence of the exact critical exponents with those used in the mean-field theory.) This is associated with the same symmetry of the systems: in both cases quantum RPT characterised by a two-component order parameter take place. Microscopically, this behaviour as mentioned above is conditioned by the fact that the Goldstone mode sound velocity tends to zero at the RPT point. Qualitatively different critical behaviour at the multicritical points characterised by $d_c = 3$ is due to the existence of the soft mode with linear dispersion law at these points.

7. Summary and conclusions

In the present paper cascade of field- or anisotropy-induced RPT in magnets with the easy-plane SA is studied and the low-temperature description of collinear phases in the cascade is offered. In particular, we calculated the renormalisation and damping of the excitation spectrum due to the anharmonicity effects, found the values of the critical fields, investigated the low-temperature thermodynamics and discussed the critical behaviour at $T = 0$. The theory is developed for the case of arbitrary spin and arbitrary relationships among the anisotropy parameter, magnetic field (perpendicular to the easy plane) and exchange constant.

It should be noted that, to date, only the simplest phases in a FM with SA of arbitrary value were investigated, namely, saturated FM phases [8, 10, 11] and non-magnetic ones [2], and now complete dynamic theories for these phases are available. Some other phases were also studied, only in the lowest (Gaussian) approximation and for a fixed S (e.g. [22] for $S = 2$, [23, 24] for $S = 3/2$, etc). The present paper can be considered as an extension of these studies in the following directions:

(i) the generalisation for an arbitrary S,

(ii) taking into account all phases where the universal description for arbitrary S can be given† and

(iii) inclusion of the anharmonicity effects in the dynamic theory.

We have demonstrated that the phase diagram and the behaviour in the whole of the magnet with single-ion anisotropy differ dramatically from that of the magnet with exchange anisotropy of the same symmetry. We should emphasise: (i) the fact that there is more than one symmetrical phase ($2S$ in the case of complete cascade) and $2S$ phases

† Non-collinear phases cannot be treated in universal manner because one should make the unitary transformations of the SU($2S + 1$) group, specific for each S, when constructing such a theory [6].

(instead of one) with spontaneously broken symmetry for the Hamiltonian with single non-trivial symmetry transformation, $\exp(i\varphi S^z)$; and (ii) the alteration of spontaneous breaking and spontaneous restoration of symmetry that takes place when the magnetic field changes monotonically. These peculiarities are connected with the following circumstances. In the presence of the single-ion anisotropy (which is the local tensorial interaction) the state of the system is described by operators that belong to the Lie algebra SU(n) ($n = 2S + 1$) (see for example [6]) characterised by rank $r = n - 1 > 1$ instead of the Lie algebra SU(2) ($r = 1$) in the case of (vector) exchange anisotropy. The rank of the algebra just defines the number of possible symmetric and non-symmetric phases in the system under consideration.

As to the peculiarities of each phase, it was interesting to trace the transformation of a properties of the phases in the cascade with the transformation of the spin-order structure from full ferromagnetic order for $m = S$ to full tensorial order for $m = 0$. It was found that the closer the phase was to the full ferromagnetic one (the lesser $S - m$) the more adequate was the harmonic description at low T. The effects of anharmonicity are intensified with growing difference $S - m$, i.e. with decreasing m at fixed S, the anharmonic corrections achieve maximum values in a non-magnetic tensorial phase, for which we have evaluated them quantitatively at $T = 0$. The mentioned dependence on $S - m$ leads in particular to unusual behaviour such as the growth of relative anharmonic corrections with growing S for a phase with fixed m (except for the case $m = S$), for example for the tensorial phase. The relative values of anisotropy and field in MP increase with growing S, and the region of existence of all collinear phases with $m = S$ shifts to infinity with $S \to \infty$; thus we reach the classical limit. So the existence of the phases under consideration is especially a quantum effect.

As to the temperature laws for the various physical characteristics for each collinear phase, these are certainly predetermined by their symmetry. Just preservation of symmetry relative to the transformation $\exp(i\varphi S^z)$ leads to the existence of an energy gap for the low-lying mode, and consequently to the dependences characterised by Z functions (see for example equations (32) and (51)) for the region far from the RPT points and the half-integer power dependences (see equation (34)) for the region close to the RPT points (but beyond the fluctuational region). Two remarks should be made:

(i) The signs of the temperature corrections are such that within each partially ordered phase area there are two ranges where the system properties differ qualitatively: the range $H > \Delta_2$ where the properties are similar to that of the disordered phase with $m = 0$ (magnetisation increases with T at $h = \text{const}$, $D = \text{const}$; magnetisation and longitudinal magnetic susceptibility increase with h at $T = \text{const}$, $D = \text{const}$; etc) and the range $H < \Delta_2$ where the properties are contrary to those described above and similar to those of the ordered phase with $m = S$.

(ii) In the vicinity of the MP where the soft modes have a linear dispersion the temperature dependences transform to the integer power laws (see equations (27) and (29)).

The above-mentioned behaviour of the thermodynamic functions and the critical fields is determined by two low-energy collective modes. As was shown, two additional high-energy modes are present in the spectrum. These modes correspond to the localised excitations in the Gaussian approximation and acquire dispersion due to the interaction with low-energy quasi-particles. The existence of these modes should be manifested in the Raman spectra, ferromagnetic resonance, inelastic neutron scattering, etc.

Finally, one should also mention the peculiarities of the critical behaviour of the

system at the RPT induced by a magnetic field and single-ion anisotropy at $T = 0$, namely, the difference in the behaviour at the multicritical points $A^{(m)}$ and at all other points of the lines $H_{ci}^{(m)}(\xi)$. It is due to the different character of the soft mode dispersion law, which is linear at MP and quadratic at all other points of lines $H_{ci}^{(m)}(\xi)$.

The experimental realisation of the investigated RPT cascade is possible in magnets with a significant value of the single-ion anisotropy constant, for example in the rare-earth compounds. Here the number of RPT in the cascade and the shape of the phase diagram on the whole depends on the hierarchy of the critical fields $H_{ci}^{(m)}$ at fixed $\xi = D/2J_0$, in particular on the value of m starting from which $H_{c1}^{(m)}$ and $H_{c2}^{(m)}$ become real and positive. It follows from equations (13) and (12) that, depending on the relation between S and D/J_0, the cascade may be complete (right from $m = 0$ for integer values of S and $m = \frac{1}{2}$ for half-integer S) or partial, and may start from the collinear phase as well as the non-collinear one (on both the ascending and descending parts of $T_c(H)$ in the latter case). An increase in S or a decrease in D/J_0 causes an increase in the value of m from which $H_{ci}^{(m)}$ become real and positive values. As to the experimental data available, one should mention [25], where an apparently similar cascade of RPT was observed in compounds $AFeCl_3$ (A = Rb, Cs) and $FeSiF_6 \cdot 6H_2O$ with $S = 2$, and the experiments for the non-magnetic phase of singlet magnets, for example for $Ni(C_5H_5NO)_6$ and $Ni(NO_3)_2 \cdot 6H_2O$ with $S = 1$ [26, 27]. We did not find any experimental data on the properties of the intermediate phases with $m \neq S$.

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